Quantification and Modality

Terry Langendoen
Professor Emeritus of Linguistics
University of Arizona

Linguistics Colloquium
University of Arizona
13 Mar 2009

Appearance

• The modal operators are like disguised quantifiers.
  Fitting & Mendelsohn 1998:108
• Why?
  1. Accessibility relations of modals with possible worlds are stated using quantifiers.
  2. De re/de dicto interpretations depend on relative scope of a modal and a quantifier, analogous to the interpretations that arise when two quantifiers are present.
  3. Some modal operators (e.g. temporals like 'always' and 'sometimes') are in fact quantifiers.
• It [is] no accident that the quantifiers look very much like modals. Koslow 1992: 312
• Why?
  – Because universal and existential quantifiers, in general, are modals, not just in special cases, e.g. when they range over temporal entities.

Reality

A. Universal and existential quantifiers are modal operators, as Koslow contends.
B. Not all modal operators are quantifiers.
• In order to demonstrate A and B, some background is needed.
• I start with an account of the notion of an implication structure.

Implication structures

• An implication structure \( I \) consists of a set \( S \) (propositions or other elements) and an implication relation \( \Rightarrow \) over them.
• The relation \( \Rightarrow \) satisfies the following axioms, where \( A_1, ..., A_n, B, B_1, ..., B_m, C \) are in \( S \):
  - Projection: \( A_1, ..., A_n \Rightarrow B \Rightarrow A_k (1 \leq k \leq n) \)
  - Simplification: If \( A_1, ..., A_i, A_j, ..., A_n \Rightarrow B \), then \( A_1, ..., A_i \Rightarrow B \)
  - Permutation: If \( A_1, ..., A_n \Rightarrow B \), then \( A_{f(1)}, ..., A_{f(n)} \Rightarrow B \) for any permutation \( f \) of \( \{1, ..., n\} \)
  - Cut: If \( A_1, ..., A_n \Rightarrow B \) and \( B, B_1, ..., B_m \Rightarrow C \), then \( A_1, ..., A_n, B_1, ..., B_m \Rightarrow C \).

Additional properties of the implication relation

• In \( A_1, ..., A_n \Rightarrow B \), the \( A_j \)s are the premises (or antecedents), and \( B \) is the conclusion (or consequence).
  - Reflexivity: \( A \Rightarrow A \) (follows from Projection)
  - Transitivity: If \( A \Rightarrow B \) and \( B \Rightarrow C \), then \( A \Rightarrow C \) (follows from Cut)
  - Antisymmetry: If \( A \Rightarrow B \) and \( B \Rightarrow A \), then \( A \leftrightarrow B \).
  - Monotonicity: If \( A_1, ..., A_n \Rightarrow B \), then \( A_1, ..., A_{n'} \Rightarrow C \Rightarrow B \) (follows from Simplification and Cut).

Dual of an implication structure

• If \( I = <S, \Rightarrow> \) is an implication structure, then its dual \( \hat{I} = <S, \Rightarrow> \) is also an implication structure, where:
  - \( A_1, ..., A_n \Rightarrow B \) iff for every \( T \) in \( S \), if \( A_1 \Rightarrow T, ..., A_n \Rightarrow T \), then \( B \Rightarrow T \).
• In the single premise case, this condition reduces to:
  - \( A \Rightarrow B \) iff \( B \Rightarrow A \).
A simple implication structure \( I_4 \) with four elements

- Let \( I_4 = <S_4, \Rightarrow> \), where \( \Rightarrow \) is a truth-preserving relation and \( S_4 = \{a, b, c, d\} \) such that:
  - \( a: \) ‘it’s hot’ \( \iff \) \( \neg b \)
  - \( b: \) ‘it’s cold’ \( \iff \) \( \neg a \)
  - \( c: \) ‘it’s hot and it’s cold’ \( \iff \) \( (a \wedge \neg a) \iff \bot \)
  - \( d: \) ‘it’s hot or it’s cold’ \( \iff \) \( (a \lor \neg a) \iff T \)
- Notation:
  - \( \neg \) negation
  - \( \wedge \) conjunction
  - \( \lor \) disjunction

Diagram of \( I_4 \), omitting reflexive arcs and arcs derivable from transitivity

Diagram of \( I_4 \), dual of \( I_4 \), in which \( \Rightarrow \) preserves falsity

Modal operators

- Next, I define what a modal operator is, following Koslow 1992, providing a purely ‘structural’ account, without using the notion of ‘possible worlds’. There are two types.
  - Necessity, represented by \( \square \) ‘box’.
  - Possibility, represented by \( \lozenge \) ‘diamond’.

Defining properties of a \( \square \) modal

- \( \square \) is a unary operator that preserves implication in an implication structure \( I \).
  - If \( A_1, \ldots, A_n \Rightarrow B \), then \( \square A_1, \ldots, \square A_n \Rightarrow \square B \), for all \( A_1, \ldots, A_n, B \) in \( S \).
- \( \square \) fails to preserve implication in the dual structure \( \bar{I} \).
  - There are \( A_1, \ldots, A_n, B \) in \( S \) such that \( A_1, \ldots, A_n \Rightarrow B \), but \( \square A_1, \ldots, \square A_n \Rightarrow \square B \) fails.

Distribution of \( \square \) with \( \wedge \) and \( \lor \)

- For the structures of interest to us, the following distribution principles are consequences of the defining properties of \( \square \).
  - DAX (“distribution of \( \wedge \) with box”)
    \( \square (A \wedge B) \iff \square A \wedge \square B \) for all \( A, B \) in \( S \).
  - NOX (“distribution of \( \lor \) with box”)
    \( \square (A \lor B) \Rightarrow \square A \lor \square B \) fails for some \( A, B \) in \( S \).
  - Selecting \( B \iff \neg A \), the premise is generally true and the consequent false.
Defining properties of a ◊ modal

- ◊ is a unary operator that fails to preserve entailment in I.
- There are A₁, ..., Aₙ, B in S such that A₁, ..., Aₙ ⇒ B, but ◊A₁, ..., ◊Aₙ ⇒ ◊B fails.
- ◊ preserves entailment in Î.
- If A₁, ..., Aₙ ⇒ B, then ◊A₁, ..., ◊Aₙ ⇒ ◊B, for all A₁, ..., Aₙ, B in S.
  - In Î, ◊ is a necessity modal and ◊ is a possibility modal.

Distribution of ◊ with ∧ and ∨

- For the structures of interest to us, the following distribution principles are consequences of the defining properties of ◊.
  - NAD ("nondistribution of and with diamond")
    ◊(A ∧ B) ⇒ ◊A ∧ ◊B fails for some A, B in S.
  - See comment on NOX.
  - DOD ("distribution of or with diamond")
    ◊(A ∨ B) ⇔ ◊A ∨ ◊B for all A, B in S.

Interdefinability of □ and ◊

- □ and ◊ are interdefinable using negation in all implication structures of interest here; i.e. for all A in S:
  □A ⇔ ¬◊¬A
  ◊A ⇔ ¬□¬A

Alethic and epistemic modality in I₄

- Next, I illustrate two types of modality in the structure I₄, alethic (only analytic statements are necessary) and epistemic (both analytic and known synthetic statements are necessary).
  - Let □ₐ and ◊ₐ represent alethic necessity and possibility.
    □ₐT = T, otherwise □ₐA = ⊥
    ◊ₐ⊥ = T, otherwise ◊ₐA = ⊥
  - Let □ₑ and ◊ₑ represent epistemic necessity and possibility.
    □ₑA = A if A is known, otherwise □ₑA = ⊥
    ◊ₑA = T if A is known, otherwise ◊ₑA = A

Diagram of alethic modality in I₄

- □ and ◊ in I₄ have the requisite distribution properties
  - DAX
    □(A ∧ B) ⇔ □A ∧ □B
    □(a ∨ b) ⇔ □a ∨ □b
  - NOX
    □A ∨ □B ⇔ □(A ∨ B)
  - NAD
    □(A ∧ B) ⇔ □A ∧ □B
    □(a ∨ b) ⇔ □a ∨ □b
  - DOD
    □(a ∨ b) ⇔ □a ∨ □b

- Similarly for all other pairs of elements of S₄.
Diagram of epistemic modality in $I_4$

\[
\begin{align*}
d: &\ 0_a \Leftrightarrow 0_d \\
&\ \Leftrightarrow 0_d \Leftrightarrow T \\
\end{align*}
\]

Additional properties of alethic and epistemic modals in $I_4$

- For all $A$ in $S_4$:
  - $\Box_a A \Rightarrow \Box_a A \Rightarrow A \Rightarrow \Box_b A \Rightarrow \Box_b A$
  - $\Box_a A \Leftrightarrow \Box_a A \Leftrightarrow \Box_a A$
  - $\Box_a A \Rightarrow \Box_a A \Rightarrow \Box_a A$
  - $\Box_a A \Rightarrow \Box_b A \Rightarrow \Box_a A$
  - $\Box_a A \Rightarrow \Box_a A \Rightarrow A \Rightarrow \Box_a A$
- However:
  - $\Box_a A \Rightarrow \Box_a A \Leftrightarrow A \Rightarrow \Box_a A$ fail for $A = b$.

Diagram of epistemic modality in $I_{10}$

\[
\begin{align*}
d: &\ T \\
g: &\ 0_a \\
h: &\ 0_b \\
a: &\ 0_a \\
b: &\ 0_a \Leftrightarrow 0_b \\
e: &\ 0_a \Leftrightarrow 0_c \\
f: &\ 0_b \\
c: &\ \perp
\end{align*}
\]

Epistemic modals in $I_4$ also have the requisite distribution properties

- DAX
  - $\Box_a (a \vee b) \Leftrightarrow \Box_a a \vee \Box_a b \Rightarrow c$.
  - Similarly for all other pairs of elements of $S_4$.
- NOX
  - $\Box_a (a \wedge b) \Leftrightarrow \Box_a a \wedge \Box_a b \Rightarrow a$; $d \Rightarrow a$ fails.
- NAD
  - $\Box_a a \wedge \Box_a b \Rightarrow b$; $\Box_a (a \wedge b) \Rightarrow c$; $b \Rightarrow c$ fails.
- DOD
  - $\Box_a (a \vee b) \Rightarrow \Box_a a \vee \Box_a b \Rightarrow d$.
  - Similarly for all other pairs of elements of $S_4$.

A linguistically realistic epistemic structure $I_{10}$

- Let $I_{10}$ be the structure $<S_{10}, \Rightarrow>$, where $S_{10} = S_4 \cup \{e, f, g, h, i, j\}$, such that:
  - $e$: ‘it must be hot’ $\Leftrightarrow \Box_a a$
  - $f$: ‘it must be cold’ $\Leftrightarrow \Box_b b$
  - $g$: ‘it might be hot’ $\Leftrightarrow \Box_a a$
  - $h$: ‘it might be cold’ $\Leftrightarrow \Box_b b$
  - $i$: ‘it must be hot or it must be cold’ $\Leftrightarrow \Box_a a \vee \Box_b b$
  - $j$: ‘it might be hot and it might be cold’ $\Leftrightarrow \Box_a a \wedge \Box_b b$

Additional properties of epistemic modals in $I_{10}$

- $\Box_a e \Leftrightarrow \Box_a f \Leftrightarrow \Box_a i \Leftrightarrow \Box_a j \Leftrightarrow c$; $\Box_a g \Rightarrow a$; $\Box_a h \Rightarrow b$
- $\Box_a e \Leftrightarrow \Box_a h \Rightarrow \overline{\Box_a i} \Leftrightarrow \Box_a i \Leftrightarrow \overline{\Box_a j} \Rightarrow d$; $\overline{\Box_a e} \Rightarrow a$; $\overline{\Box_a f} \Rightarrow b$
- Consequently for all $A$ in $S_{10}$:
  - $\Box_a \Box_a A \Rightarrow \Box_a \Box_a A$
  - $\Box_a \Box_a A \Rightarrow \Box_a A$, but not conversely
  - $\Box_a A \Rightarrow \Box_a \Box_a A$, but not conversely
  - $A \Rightarrow \Box_a \Box_a A$ (recall that this fails in $I_4$)
- However:
  - $\Box_a A \Rightarrow \Box_a \Box_a A$ fails for $A = a$ and $A = b$. 

Terry Langendoen
Finally we get to quantifiers!
But first, ...

• The analysis of quantifiers requires the use of extended implication structures \( I = \langle E, \Pi, S, \Rightarrow \rangle \), where \( E \) is a set of entities, \( \Pi \) a set of predicates (open sentences), \( S \) a set of (closed) sentences, and \( \Rightarrow \) a truth-preserving implication relation. For ease of presentation, I consider only one-place predicates.

Predicate implication and universal quantification \( \forall ' \text{for all}' \)

• Predicate implication \( \Rightarrow^* \) is defined in terms of sentential implication as follows. For all \( P_1, ..., P_n \) in \( \Pi \), \( P_1, ..., P_n \Rightarrow^* Q \) iff \( P_i(e_i), ..., P_n(e_n) \Rightarrow Q(f) \) for all choices of \( e_i, ..., e_n, f \) in \( E \).
• That is (where \( \forall P \) represents \( \forall xP \), etc.):

Unifying predicate and sentential implication

• Extending sentential implication to also cover predicate implication (so that both open and closed sentences can appear in the same implication, and so that the implication relation over open sentences is the same as that over closed ones), we have:

Confirming that \( \forall \) is a necessity modal

• \( \forall \) satisfies DAX. For all predicates \( P, Q \) in \( \Pi \):
  \( \forall P \land Q \Leftrightarrow \forall (P \land Q) \)
  
  – This is a well known property of universal quantification.
• \( \forall \) satisfies NOX. For some predicates \( P, Q \) in \( \Pi \):
  \( \forall (P \lor Q) \Rightarrow \forall P \lor \forall Q \)
  
  – If neither \( P \) nor \( Q \) holds for every member of \( E \) but their disjunction does, then the premise is true and the consequent false.

The existential quantifier \( \exists ' \text{for some}' \) is a possibility modal

• \( \exists \) satisfies NAD. There are predicates \( P, Q \) in \( \Pi \) such that:
  \( \exists P \land Q \Rightarrow \exists (P \land Q) \) fails.
  
  – If \( P \) and \( Q \) hold for different members of \( E \), then the premise is true and the consequent false.
• \( \exists \) satisfies DOD. For all predicates \( P, Q \) in \( \Pi \):
  \( \exists (P \lor Q) \Leftrightarrow \exists P \lor \exists Q \)
  
  – This is a well known property of existential quantification.

Interdefinability of \( \forall \) and \( \exists \)

• \( \forall \) and \( \exists \) are interdefinable using negation, just like \( \Box \) and \( \Diamond \). For all \( P \) in \( \Pi \):
  \( \forall P \Leftrightarrow \neg \exists \neg P \)
  \( \exists P \Leftrightarrow \neg \forall \neg P \)
A simple quantificational structure $I_8$

- Let $I_8$ be the extended structure $<E, \Pi_2, S_8, \Rightarrow>$, where
  - $E$ is a plural set of people, $\Pi_2 = \{H \ 'happy', \ U \ 'unhappy')$,
  - $S_8 = \{k, l, m, n, o, p, q, r\}$,
  - $\Rightarrow$ is a truth-preserving relation, and:
    - k: $H \iff \forall H$ 'everyone is happy'
    - l: $U \iff \forall U$ 'everyone is unhappy'
    - m: $\forall H \land \forall U \iff \bot$ 'everyone is happy and everyone is unhappy'
    - n: $\exists H \lor \exists U \iff T$ 'someone is happy or someone is unhappy'
    - o: $\exists H$ 'someone is happy'
    - p: $\exists U$ 'someone is unhappy'
    - q: $\forall H \lor \forall U$ 'everyone is happy or everyone is unhappy'
    - r: $\exists H \land \exists U$ 'someone is happy and someone is unhappy'

Diagram of quantificational modality in $I_8$

Reality check A

A. The principle that universal and existential quantifiers are modal operators is now established.
- Quantified modal logic is simply modal logic, which includes the modal quantifiers $\forall$, $\exists$, and (as shown below) others in its purview.
- In natural languages, modals are expressed not only by tense, mood and aspect markers and adverbials, as Fitting & Mendelsohn 1998: 2 contend, but also by quantifiers that reduce the placedness of predicates.
- Teaser question: Might there also be modals that increase the placedness of predicates?

Reality check B

B. The principle that not all modal operators are quantifiers is established by observing that the alethic and epistemic modals we have considered are not quantifiers.
- However, the appearances noted by Fitting & Mendelsohn are not deceiving.
  - Classification of modals according to properties of their accessibility relations can be accounted for without using possible-world models; see Koslow 1992: chapters 34-35.
  - De re/de dicto interactions follow from the general modal law $\Diamond \Box \Rightarrow \Box \Diamond$, from which we obtain $\exists \Box \equiv \Box \exists$ and $\forall \Box \iff \Box \forall$, as special cases; also, for multiple quantification, $\exists \forall \Rightarrow \forall \exists$.

Other modal quantifiers

- $\forall$ and $\exists$ are not the only modal quantifiers.
- Let $I_z = <E_z, \Pi, S_z, \Rightarrow>$ be an extended implication structure in which $E_z$ is the set of positive integers and $\Pi$ is a set of one-place predicates over $E_z$.
- Then $\forall f$ and $\exists x$ are modal quantifiers in $I_z$ where:
  - $\forall f$ represents 'for all but at most finitely many members of $E_z$'.
  - $\exists x P$ represents 'for infinitely many members of $E_z$'.

$\forall f$ is a necessity modal

- $\forall f$ satisfies DAX.
  - $\forall f(P \land \forall fQ) \Rightarrow \forall f(P \land Q)$ for all $P, Q$ in $\Pi$.
  - This equivalence follows from the fact that the union of finite sets is finite.
- $\forall f$ satisfies NOX.
  - $\forall f(P \land Q) \Rightarrow \forall fP \land \forall fQ$ fails for some $P, Q$ in $\Pi$.
  - If $Q = \neg P$, where $P$ and $Q$ each hold for infinitely many positive integers, then the premise is true and the consequent false.
\( \exists^\infty \) is a possibility modal

- \( \exists^\infty \) satisfies NAD.
  \( \exists^\infty P \land \exists^\infty Q \Rightarrow \exists^\infty (P \land Q) \) fails for some \( P, Q \) in \( \Pi \).
  - Again, if \( Q = \neg P \), where \( P \) and \( Q \) each hold for infinitely many positive integers, then the premise is true and the consequent false.
- \( \exists^\infty \) satisfies DOD.
  \( \exists^\infty (P \lor Q) \iff \exists^\infty P \lor \exists^\infty Q \) for all \( P, Q \) in \( \Pi \).
  - This equivalence also follows from the fact that the union of finite sets is finite.

Other quantifiers are not modal

- Let \( I_2 = < E_1, \Pi, S_2, \Rightarrow > \) be an extended implication structure in which \( E_1 \) is large. If \( E_1 \) is finite, let \( n = \text{card}(E_1) \), \( k = n/2 \) for even \( n \), and \( k = (n-1)/2 \) for odd \( n \).
- Then none of the following quantifiers are modal:
  \( \forall^1 \) 'for all but at most 1'
  \( \exists^2 \) 'for at least 2'
  ... 
  \( \forall^k \) 'for all but at most \( k \)'
  \( \exists^{k+1} \) 'for at least \( k+1 \)'
  ... 
  \( \forall^w \) 'for most' = 'for all but at most a few'
  \( \exists^m \) 'for at least many'

Why \( \forall^* \) and \( \exists^* \) preserve one-premise implications

- If \( P \Rightarrow Q \), then \( \{x|Px\} \subseteq \{y|Qy\} \).
  Consequently, \( \forall^* P \Rightarrow \forall^* Q \) and \( \exists^* P \Rightarrow \exists^* Q \).
- Counter-models showing that \( \forall^1 \) and \( \exists^2 \) fail to preserve multi-premise implications are provided in the next slide; similar models can be provided for all the other members of \( \forall^* \) and \( \exists^* \).

Why \( \forall^* \) and \( \exists^* \) fail to preserve multi-premise implications

- Let \( P, Q \Rightarrow P \land Q \) for all \( P, Q \) in \( \Pi \). However for some \( P, Q \) in \( \Pi \):
  \( \forall^1 P, \forall^1 Q \Rightarrow \forall^1 (P \land Q) \) fails.
  \( \exists^2 P, \exists^2 Q \Rightarrow \exists^2 (P \land Q) \) fails.
  - In the first case, if \( P \) and \( Q \) fail to hold for different single members of \( E \) only, then the premise is true but the consequent is false.
  - In the second case, if \( P \) and \( Q \) hold for distinct pairs of members of \( E \) only, then the premise is true but the consequent is false.
Quasimodal operators

- Let us call unary operators that preserve one-premise implications but not multi-premise ones quasimodals.
- Symbolizing quasimodals by $\diamondsuit$, we have:
  1. If $A \rightarrow B$, then $\diamondsuit A \rightarrow \diamondsuit B$ for all $A, B$ in $S$.
  2. However, there are $A_1, ... A_n, B$ in $S$ such that $A_1, ... A_n \rightarrow B$, but $\diamondsuit A_1, ... \diamondsuit A_n \rightarrow \diamondsuit B$ fails.
- Similarly for $\hat{\cdot}$.

Quasimodal quantifiers

- $\forall^*$ and $\exists^*$ are all quasimodals, more precisely, quasimodal quantifiers, that satisfy the requirement:
  If $P \Rightarrow Q$, then $\diamondsuit P \Rightarrow \diamondsuit Q$ for all $P, Q$ in $\Pi$.
  However, there are $P_1, ... P_n, Q$ in $\Pi$ such that $P_1, ... P_n \Rightarrow Q$, but $\diamondsuit P_1, ... \diamondsuit P_n \Rightarrow \diamondsuit Q$ fails.

Quasinecessity and quasipossibility

- Quasimodals, like modals, come in interdefinable pairs. For example:
  $\forall/1 \leftrightarrow \neg \exists^2$ and $\forall/w \leftrightarrow \exists^m$.
  $\exists^2 \leftrightarrow \neg \forall/1$ and $\exists^m \leftrightarrow \neg \forall/w$.

Comparing the modals $\forall/f$ and $\exists^\infty$ with the quasimodals $\forall/w$ and $\exists^m$

- The diagram of $I_\ell$ including $\forall$, $\forall/w$, $\exists$, $\exists^m$ has exactly the same form as the diagram of $I_\ell$ including $\forall$, $\forall/f$, $\exists$, $\exists^\infty$, even though $\forall/f$ and $\exists^\infty$ are modal and $\forall/w$ and $\exists^m$ are quasimodal.

Non-quantifier quasimodals

- In a large enough implication structure in which the epistemic modals $\Box_\varepsilon$ and $\Diamond_\varepsilon$ are defined, epistemic quasinecessity and quasipossibility operators $\Box_\varepsilon$ and $\Diamond_\varepsilon$ can also be defined such that for all $A$ in $S$:
  $\Box_\varepsilon A \Rightarrow \Box_\varepsilon A \Rightarrow \Diamond_\varepsilon A \Rightarrow \Diamond_\varepsilon A$
  analogous to $\forall^*$ and $\exists^*$ in relation to $\forall$ and $\exists$.
  - In such a structure $\Box_\varepsilon$ may be expressed by ‘probably’ and $\diamondsuit_\varepsilon$ as ‘not improbably’. 

Diagram of $I_\ell$ including $\forall$, $\forall/w$, $\exists$, $\exists^m$
‘Probably’ and ‘not improbably’ are quasimodals, not modals

- The operator $\nabla_v$ ‘probably’ is a quasimodal, since it preserves one-premise implications, but not multi-premise ones, in a rich enough epistemic structure.
  - The answer to Fitting & Mendelsohn’s 1998: 3, Exercise 1.1.1.1: Is ‘probably’ a modal? is quasipositive!
- Also, since $\Delta_v$ ‘not improbably’ $\iff \neg \nabla_v \neg$, it too is a quasimodal in that structure.
- Finally, since $\nabla_v \Rightarrow \Delta_v$, the former is a quasinecessity and the latter a quasipossibility.

References