Modals and Quasimodals

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Overview: Modals

- In the first part of this talk I present a "structuralist" account of modal (necessity and possibility) operators that characterizes them exclusively in terms of their roles in entailment.
  - The discussion is based on Arnold Koslow, *A Structuralist Theory of Logic*, Cambridge University Press, 1992.
  - It includes consideration of epistemic, deontic and quantificational modals, and an exploration of their interactions.

Overview: Quasimodals

- In the second part, I provide a Koslow-style analysis of "quasimodal" (quasinecessity and quasipossibility) operators that are distinct from but are closely related to modals.
- I define several types of quasimodals and explore their interactions with each other and with true modals.

Modals

1. Entailment structures
2. Modal essentials
3. Varieties of modality
4. Interactions between varieties

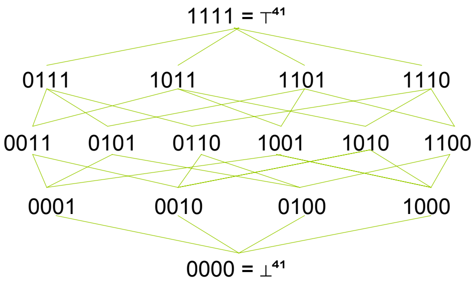
Definition of entailment structure

- E = <S, $\models$> is an entailment structure, where S is a set and $\models$ is an entailment relation over S.
- $\models$ obeys standard (Gentzen) axioms for reasoning:
  1. Projection (Reflexivity is a special case.)
  2. Simplification (Repetition of premises may be eliminated.)
  3. Permutation (Order of premises doesn't matter.)
  4. Dilution (Thinning, Monotonicity – follows from Simplification and Cut)
  5. Cut (Transitivity is a special case.)

Example of an entailment structure

- Let S^4 be the set of bit strings of length 4 = {0000, 0001, ..., 1110, 1111}.
- Let $\models^1$ be the relation that preserves 1 in each of the four positions in the bit string:
  - $p_1, \ldots p_n \models^1 q$ iff whenever every p_i ($1 \leq i \leq n$) has 1 in some position, then so does q.
- Then E^{41} = <S^4, $\models^1$> is an entailment structure that can be diagrammed as in H^{41} on the next slide, where the arcs, representing one-premise entailments, are to be read upward (reflexive arcs omitted). $\top$ represents "top" and $\bot$ "bottom".

Hasse diagram H^{41} of E^{41}



$E^{40} = \hat{E}^{41}$, the dual of E^{41}

- H^{41} can also be interpreted as representing $E^{40} = \hat{E}^{41}$, the dual of E^{41}, by reading the arcs downwards, e.g. 1010 $\models^{0}$ 1000.
- The dual entailment relation $\models^{0}$ preserves 0 in each position in the bit string.
  - $p_1, \ldots p_n \models^{0} q$ iff whenever every p_i ($1 \leq i \leq n$) has 0 in some position, then so does q.
  - 0000 = $\top^{40}$ and 1111 = $\bot^{40}$.

Defining properties of modals

$\Box$ Necessity (“box”)

1. $\Box$ preserves entailment in E. For all $p_1, \ldots, p_n, q \in S$:
   - if $p_1, \ldots, p_n \models q$, then $\Box p_1, \ldots, \Box p_n \models \Box q$
2. $\Box$ does not preserve entailment in $\hat{E}$ (dual of E). For some $p_1, \ldots, p_n, q \in S$:
   - if $p_1, \ldots, p_n \models^{\wedge} q$, then $\Box p_1, \ldots, \Box p_n \not\models^{\wedge} \Box q$.

$\Diamond$ Possibility (“diamond”)

1. $\Diamond$ preserves entailment in $\hat{E}$. For all $p_1, \ldots, p_n, q \in S$:
   - if $p_1, \ldots, p_n \models^{\wedge} q$, then $\Diamond p_1, \ldots, \Diamond p_n \models^{\wedge} \Diamond q$.
2. $\Diamond$ does not preserve entailment in E. For some $p_1, \ldots, p_n, q \in S$:
   - if $p_1, \ldots, p_n \models q$, then $\Diamond p, \Diamond q \not\models \Diamond r$.

Interdefinability of $\Box$ and $\Diamond$

- $\Diamond$ is the *dual* of $\Box$, so is definable in terms of $\Box$ and negation as follows:
  - $\Diamond p \Leftrightarrow \neg\Box\neg p$
- Similarly, $\Box$ is the *dual* of $\Diamond$, and is definable in terms of $\Diamond$ and negation as follows:
  - $\Box p \Leftrightarrow \neg\Diamond\neg p$

Modals in E^{41}

- $\Box^{41}$ is defined for E^{41} as follows.
  1. $\Box^{41}p = \top^{41}$ if $p = \top^{41}$
  2. $\Box^{41}p = \bot^{41}$ otherwise
- $\Diamond^{41}$ is defined for E^{41} as follows.
  1. $\Diamond^{41}p = \bot^{41}$ if $p = \bot^{41}$
  2. $\Diamond^{41}p = \top^{41}$ otherwise

$\Box^{41}$ and $\Diamond^{41}$ satisfy definitions $\Box$1 and $\Diamond$1

- $\Box^{41}$ preserves entailment in E^{41}.
  - Suppose $p_1, \ldots, p_n \models^{1} q$, where $q = \top^{41}$.
    - Since $\Box^{41}q = \top^{41}$, $\Box^{41}p_1, \ldots, \Box^{41}p_n \models^{1} \Box^{41}q$ for every $p_1, \ldots, p_n \in S^4$
  - Otherwise, q and at least one of the premises p_i contain 0 in a certain position, so that $\Box^{41}p_i$ and $\Box^{41}q = \bot^{41}$.
    - $\Box^{41}p, \ldots, \Box^{41}p_n \models^{1} \Box^{41}q$ follows from the projection axiom.
- $\Diamond^{41}$ preserves entailment in E^{40}.
  - The arguments are similar.

$\Diamond^{41}$ and $\Box^{41}$ satisfy definitions $\Diamond 2$ and $\Box 2$

- $\Diamond^{41}$ does not preserve entailment in E^{41}.
  - 0111, 1000 $\models^1 \bot^{41}$
  - Since $\Diamond^{41}0111 = \Diamond^{41}1000 = \top^{41}$, and $\Diamond^{41}\bot^{41} = \bot^{41}$, then $\Diamond^{41}0111, \Diamond^{41}1000 \not\models^1 \Diamond^{41}\bot^{41}$
- $\Box^{41}$ does not preserve entailment in E^{40}.
  - The argument is similar.

$\Diamond$-over-$\Box$: A special modal law

- Certain entailment structures support modal systems that obey additional "special" modal laws, while others do not. One such special law is:

1. For all p $\in$ S: $\Diamond p \models \Box\Diamond p$
   - From (1) and the general modal law $\Diamond\Box p \models \Diamond p$, it follows from the transitivity of $\models$ that:
2. For all p $\in$ S: $\Diamond\Box p \models \Box\Diamond p$
   - Let us call (2) the "$\Diamond$-over-$\Box$" law.

$\Box^{41}$ and $\Diamond^{41}$ obey $\Diamond$-over-$\Box$

- To show that $\Box^{41}$ and $\Diamond^{41}$ obey $\Diamond$-over-$\Box$, we need to consider three cases.
  1. p = $\top^{41}$: $\Diamond$-over-$\Box$ reduces to: $\top^{41} \models^{41} \top^{41}$.
  2. p = $\bot^{41}$: $\Diamond$-over-$\Box$ reduces to: $\bot^{41} \models^{41} \bot^{41}$.
  3. $\bot^{41} < p < \top^{41}$: $\Diamond$-over-$\Box$ reduces to: $\bot^{41} \models^{41} \top^{41}$.
- Next we consider two modal systems of linguistic interest that obey $\Diamond$-over-$\Box$, and one that does not.

Epistemic modals

- Let $E^{\pi e}$ = <S^π, $\models^e$> be an entailment structure where S^π is a set of propositions and $\models^e$ supports reasoning with epistemic modals.
  - Epistemic necessity ($\Box^e$):
    - $\Box^e p$ is true iff p $\in S^\pi$ is certain.
  - Epistemic possibility ($\Diamond^e$):
    - $\Diamond^e p$ is false iff $\neg p \in S^\pi$ is certain.

$\Box^e$ and $\Diamond^e$ satisfy the definitions of $\Box$ and $\Diamond$

- Demonstration is similar to that for $\Box^{41}$ and $\Diamond^{41}$ in slides 13 and 14.
- Note that if the set of logically necessary ($\Box^l$) propositions (the set for which $\Box^l p$ maps to $\top^e$) is a subset of the epistemically necessary ones, then $\Box^l p \models^e \Box^e p \models^e \Diamond^e p \models^e \Diamond^l p$, i.e. $\Box^e$ and $\Diamond^e$ are "between $\Box^l$ and $\Diamond^l$.

$\Box^e$ and $\Diamond^e$ also obey $\Diamond$-over-$\Box$

- There are three cases to consider:
- p is certain:
  - $\Diamond^e\Box^e p \models^e \Box^e\Diamond^e p$ holds because both premise and conclusion are true.
- ¬p is certain:
  - $\Diamond^e\Box^e p \models^e \Box^e\Diamond^e p$ holds because both premise and conclusion are false.
- Otherwise:
  - $\Diamond^e\Box^e p \models^e \Box^e\Diamond^e p$ holds because the premise is false while the conclusion is true.

Quantifier modals

- Let $E^{nq} = \langle S^n, I, \models^q \rangle$ be a first-order (extended) entailment structure, where S^n is a set of sentences with up to *n* open places ($n \geq 0$), and I is a set of individuals with at least two members.
  - The universal quantifier $\forall x_i$ is a necessity modal $\Box^q$, mapping from S^n to S^{n-1} ($1 \leq i \leq n$).
  - The existential quantifier $\exists x_i$ is a possibility modal $\Diamond^q$, mapping from S^n to S^{n-1} ($1 \leq i \leq n$).

$\forall$ and $\exists$ are interdefinable using negation, and also obey $\Diamond$-over-$\Box$

- Interdefinability of $\forall$ and $\exists$ using negation
  - $\forall x Px \Leftrightarrow^q \neg\exists x \neg Px$
  - $\exists x Px \Leftrightarrow^q \neg\forall x \neg Px$
- The principle $\exists x \forall y Pxy \models^q \forall y \exists x Pxy$ instantiates $\Diamond$-over-$\Box$ for first-order logic.
  - *there is some girl who likes every boy* $\models^q$ *every boy is such that some girl likes him*

Deontic modals

- Let $E^{\alpha d} = \langle S^\alpha, I, \models^d \rangle$ be an (extended) entailment structure, where S^α is a set of open action sentences, I a set of individuals capable of acting as agents, and $\models^d$ supports reasoning with deontic modals.
  - Deontic necessity ($\Box^d$):
    - $\Box^d Pa$ is true iff $a \in I$ is required to do $P \in S^\alpha$.
  - Deontic possibility ($\Diamond^d$):
    - $\Diamond^d p(a)$ is false iff $a \in I$ is required not (not permitted) to do $P \in S^\alpha$.

$\Diamond^d$ and $\Box^d$ do not obey $\Diamond$-over-$\Box$

- $\Diamond^d \Box^d Pa \Leftrightarrow^d \Diamond^d Pa$ (cf. slide 15)
  - *Ann is permitted to be required to leave* $\Leftrightarrow^d$ *Ann is permitted to leave*
- $\Box^d \Diamond^d Pa \models^d \Diamond^d Pa$
  - *Ann is required to be permitted to leave* $\models^d$ *Ann is permitted to leave*
- Hence $\Box^d \Diamond^d Pa \models^d \Diamond^d \Box^d Pa$, which is $\Box$-over-$\Diamond$, the converse of $\Diamond$-over-$\Box$.

Epistemic and quantifier modal interactions obey $\Diamond$-over-$\Box$

- $\Diamond^e$ with $\forall$
  - $\Diamond^e \forall x Px \models^q \forall x \Diamond^e Px$
    - *it is possible that every candidate will win* $\models^q$ *for every candidate it is possible that he or she will win*
    - more pithily: *every candidate might win* $\models^q$ *any candidate might win*
- $\exists$ with $\Box^e$
  - $\exists x \Box^e Px \models^q \Box^e \exists x Px$
    - *there is a candidate who it is certain will win* $\models^q$ *it is certain that a candidate will win*

Deontic and quantifier modal interactions also obey $\Diamond$-over-$\Box$

- Case 1: Quantifier binds obligatee or permitee.
  - $\Diamond^d$ with $\forall$: $\Diamond^d \forall x Px \Leftrightarrow^q \forall x \Diamond^d Px$
    - *it is permitted for every boy to leave* $\models^q$ *for every boy it is permitted for him to leave*
    - or: *every boy can leave* $\models^q$ *any boy can leave*
  - $\exists$ with $\Box^d$: $\exists x \Box^d Px \Leftrightarrow^q \Box^d \exists x Px$
    - *there is a boy who is required to leave* $\models^q$ *it is required for a boy to leave*

Deontic and quantifier modal interactions also obey $\Diamond$-over-$\Box$

- Case 2: Quantifier does not bind obligatee or permitee.
  - $\Diamond^d$ with $\forall$: $\Diamond^d\forall x Pax \models^q \forall x \Diamond^d Pax$
    - *it is permitted for Ann to tutor every girl* $\models^q$ *for every girl it is permitted for Ann to tutor her*
    - or: *Ann can tutor every girl* $\models^q$ *Ann can tutor any girl*
  - $\exists$ with $\Box^d$: $\exists x \Box^d Pax \models^q \Box^d \exists x\, Pax$
  - $\exists x \Box^d Pax \models^q \Box^d \exists x\, Pax$
    - *there is a girl who Ann is required to tutor* $\models^q$ *it is required for Ann to tutor a girl*

Deontic and epistemic modals do not obey $\Diamond$-over-$\Box$

- $\Diamond^e\Box^d Pa \not\models^d \Box^d\Diamond^e Pa$, and conversely
  - *it is possible that Ann is obligated to leave* $\not\models^d$ *Ann is obligated for it to be possible for her to leave*, and conversely.
- $\Diamond^d\Box^e Pa \not\models^d \Box^e\Diamond^d Pa$, and conversely
  - *Ann is permitted for it to be certain for her to leave* $\not\models^d$ *it is certain that Ann is permitted to leave*, and conversely.

Quasimodals

1. Quasimodal essentials
2. Varieties of quasimodals
3. Quasimodal interactions
4. Quasimodal–modal interactions

Defining properties of quasimodals

⌂ An operator ⌂ ("house") is a quasimodal if and only if the following four conditions are satisfied.

1. ⌂ is intermediate in strength between $\Box$ and $\Diamond$.
   - For all $p \in S$: $\Box p \models \text{⌂}p \models \Diamond p$
2. ⌂ distributes over all single-premise implications.
   - For all $p, q \in S$: if $p \models q$, then $\text{⌂}p \models \text{⌂}q$.
3. ⌂ fails to distribute over some multi-premise implications.
   - For some $p, q, r \in S$: $p, q \models r$, but $\text{⌂}p, \text{⌂}q \not\models \text{⌂}r$.
     - Note: this condition distinguishes ⌂ from $\Box$.
4. ⌂ fails to distribute over some disjunctions.
   - For some $p, q \in S$: $\text{⌂}(p \vee q) \not\models \text{⌂}p \vee \text{⌂}q$
     - Note: this condition follows from the second part of the definition of $\Box$, thus distinguishing ⌂ from $\Diamond$.

Quasinecessity

- A quasimodal is a *quasinecessity* ($^{\Box}\text{⌂}$) operator iff for all $p \in S$:
  - $^{\Box}\text{⌂}p \models \neg{}^{\Box}\text{⌂}\neg p$, and not conversely
- From this condition, it follows that quasinecessity distributes over the law of contradiction. For all $p, q \in S$:
  - $^{\Box}\text{⌂}p, {}^{\Box}\text{⌂}\neg p \models {}^{\Box}\text{⌂}q$

Quasipossibility

- *Quasipossibility* ($^{\Diamond}\text{⌂}$) is the dual of quasinecessity. That is, for any quasinecessity operator $^{\Box}\text{⌂}$, its dual $^{\Diamond}\text{⌂}$ is a quasipossibility operator. For all $p \in S$:
  - $^{\Diamond}\text{⌂}p \Leftrightarrow \neg{}^{\Box}\text{⌂}\neg p$
- Consequently, $^{\Box}\text{⌂}$ and $^{\Diamond}\text{⌂}$ are interdefinable using negation, just like $\Box$ and $\Diamond$.
  - $^{\Box}\text{⌂}p \Leftrightarrow \neg{}^{\Diamond}\text{⌂}\neg p$

Quasimodals in E^{41}

- □⌂^{41} is defined for E^{41} as follows.
  1. □⌂^{41}p = ⊤^{41} if p has at most one 0, i.e. if p ∈ {⊤^{41}, 0111, 1011, 1101, 1110}.
  2. □⌂^{41}p = ⊥^{41} otherwise
- ◇⌂^{41} is defined for E^{41} as follows.
  1. ◇⌂^{41}p = ⊥^{41} if p has at most one 1, i.e. if p ∈ {⊥^{41}, 0001, 0010, 0100, 1000}.
  2. ◇⌂^{41}p = ⊤^{41} otherwise
- (See slide 12 for definitions of □^{41} and ◇^{41}.)

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□⌂^{41} and ◇⌂^{41} satisfy the four quasimodal conditions

1. □⌂^{41} and ◇⌂^{41} are intermediate in strength between □^{41} and ◇^{41}.
   - □^{41}p $\vDash^{41}$ □⌂^{41}p $\vDash^{41}$ ◇⌂^{41}p $\vDash^{41}$ ◇^{41}p
2. □⌂^{41} and ◇⌂^{41} distribute over single-premise entailments. If p $\vDash^{41}$ q, then:
   - □⌂^{41}p $\vDash^{41}$ □⌂^{41}q and ◇⌂^{41}p $\vDash^{41}$ ◇⌂^{41}q
3. □⌂^{41} and ◇⌂^{41} fail to distribute over some multi-premise entailments.
   - 1110, 0111 $\vDash^{41}$ 0110; but □⌂^{41}1110, □⌂^{41}0111 $\nvDash^{41}$ □⌂^{41}0110
   - 1100, 0110 $\vDash^{41}$ 0100; but ◇⌂^{41}1100, ◇⌂^{41}0110 $\nvDash^{41}$ ◇⌂^{41}0100
4. □⌂^{41} and ◇⌂^{41} fail to distribute over some disjunctions.
   - □⌂^{41}(1100 v 0110) $\nvDash$ □⌂^{41}1100 v □⌂^{41}0110

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Other properties of □⌂^{41} and ◇⌂^{41}

- □⌂^{41} distributes over the law of contradiction. For all p, q ∈ S^{4}:
  - □⌂^{41}p, □⌂^{41}¬p $\vDash^{41}$ q
- ◇⌂^{41} is the dual of □⌂^{41}.
  - ◇⌂^{41}p $\Leftrightarrow^{41}$ ¬□⌂^{41}¬p
- ◇⌂^{41} does not distribute over the law of contradiction. Let p = 0011, q = 0001. Then:
  - ◇⌂^{41}0011, ◇⌂^{41}1100 $\nvDash^{41}$ ◇⌂^{41}0001

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Epistemic quasimodals

- Let E$^{\pi e}$ = <S$^{\pi}$, $\vDash^{e}$> be the entailment structure previously defined for epistemic modals.
  - Epistemic quasinecessity □⌂^{e}:
    - □⌂^{e}p is true iff p ∈ S$^{\pi}$ is likely (should happen).
  - Epistemic quasipossibility ◇⌂^{e}:
    - ◇⌂^{e}p is true iff p ∈ S$^{\pi}$ is not unlikely.

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□⌂^{e} and ◇⌂^{e} satisfy the four quasimodal conditions

1. The epistemic quasimodals are intermediate in strength between □^{e} and ◇^{e}.
   - □^{e}p $\vDash^{e}$ □⌂^{e}p $\vDash^{e}$ ◇⌂^{e}p $\vDash^{e}$ ◇^{e}p
   - certain(p) $\vDash^{e}$ likely(p) $\vDash^{e}$ not-unlikely(p) $\vDash^{e}$ possible(p)
2. □⌂^{e} and ◇⌂^{e} distribute over single-premise entailments. If p $\vDash^{e}$ q, then:
   - □⌂^{e}p $\vDash^{e}$ □⌂^{e}q and ◇⌂^{e}p $\vDash^{e}$ ◇⌂^{e}q

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□⌂^{e} and ◇⌂^{e} satisfy the four quasimodal conditions

3. □⌂^{e} and ◇⌂^{e} fail to distribute over some multi-premise entailments.
   - There are p, q that are likely, but such that their conjunction is not likely; and there are r, s that are not unlikely, but such that their conjunction is unlikely. That is:
     - p, q $\vDash^{e}$ p ∧ q; but □⌂^{e}p, □⌂^{e}q $\nvDash^{e}$ □⌂^{e}p ∧ q
     - r, s $\vDash^{e}$ r ∧ s; but ◇⌂^{e}r, ◇⌂^{e}s $\nvDash^{e}$ ◇⌂^{e}r ∧ s
4. □⌂^{e} and ◇⌂^{e} fail to distribute over some disjunctions.
   - There are p, q that are not likely, but such that their disjunction is likely; and there are r, s that are unlikely, but such that their disjunction is not unlikely. That is:
     - □⌂^{e}(p v q) $\nvDash^{e}$ □⌂^{e}p v □⌂^{e}q
     - ◇⌂^{e}(p v q) $\nvDash^{e}$ ◇⌂^{e}p v ◇⌂^{e}q

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Other properties of □⌂^{e} and ◇⌂^{e}

- □⌂^{e} distributes over the law of contradiction, since there is no p ∈ S^π that is both likely and not likely. That is, for every p, q ∈ S^π:
  - □⌂^{e}p, □⌂^{e}¬p ⊨^{e} q
- ◇⌂^{e}p is the dual of □⌂^{e}p.
  - ◇⌂^{e}p ⇔^{e} ¬□⌂^{e}¬p
  - not-unlikely(p) ⇔ ¬[likely(¬p)] = ¬[unlikely(p)]
- ◇⌂^{e} does not distribute over the law of contradiction, since there is a p ∈ S^π such that both it and ¬p are not unlikely. For such p:
  - ◇⌂^{e}p, ◇⌂^{e}¬p ⊭^{e} ◇⌂^{e}(p ∧ ¬p)

Quantifier quasimodals

- Let E^{iq} = <S^i, I, ⊨^{q}> be a first-order (extended) entailment structure as before, but where I contains a sufficiently large number of individuals.
  - Quantifier quasinecessity M:
    - MxPx is true iff Pi is true for all but fewer than a large number of i ∈ I.
  - Quantifier quasipossibility μ:
    - μxPx is true iff Pi is true for a large number of i ∈ I.
  - I use M and μ to label these operators rather than □⌂^{q} and ◇⌂^{q}, since I will consider other quantifier quasimodals.
  - I assume that M is expressible by *most* and μ by *many*.

M and μ satisfy the four quasimodal conditions

1. M and μ are intermediate in strength between ∀ and ∃.
   - ∀xPx ⊨^{q} MxPx ⊨^{q} μxPx ⊨^{q} ∃xPx
     - *every dog is smart* ⊨^{q} *most dogs are smart* ⊨^{q} *many dogs are smart* ⊨^{q} *some dog is smart*
2. M and μ distribute over single-premise entailments. If Px ⊨^{q} Qx, then:
   - MxPx ⊨^{q} MxQx μxPx ⊨^{q} μxQx
     - *most dogs bark* ⊨^{q} *most dogs make noise*
     - *many dogs bark* ⊨^{q} *many dogs make noise*

M and μ satisfy the four quasimodal conditions

3. M and μ fail to distribute over some multi-premise entailments. There are predicates Px, Qx such that:
   - MxPx, MxQx ⊭^{q} Mx(Px ∧ Qx)
     - *most dogs bark, most dogs bite* ⊭^{q} *most dogs bark and bite*
   - μxPx, μxQx ⊭^{q} μx(Px ∧ Qx)
     - *many dogs bark, many dogs bite* ⊭^{q} *many dogs bark and bite*
4. M and μ fail to distribute over some disjunctions. There are predicates Px, Qx such that:
   - Mx(Px ∨ Qx) ⊭^{q} MxPx ∨ MxQx
     - *most dogs bark or bite* ⊭^{q} *most dogs bark or most dogs bite*
   - μx(Px ∨ Qx) ⊭^{q} μxPx ∨ μxQx
     - *many dogs bark or bite* ⊭^{q} *many dogs bark or many dogs bite*

Other properties of M and μ

- M distributes over the law of contradiction, since MxPx and Mx¬Px are contradictory.
  - *most dogs bark and most dogs do not bark* is a contradiction.
- μxPx is the dual of MxPx.
  - μxPx ⇔^{q} ¬Mx¬Px
    - *many dogs bark* ⇔ *it is not the case for most dogs that they do not bark*
- μ does not distribute over the law of contradiction, since μxPx and μx¬Px are not contradictory.
  - *many dogs bark, many dogs do not bark* ⊭^{q} *many dogs bark and do not bark*.

Other quantifier quasipossibilities

- In addition to μ, the following are all quantifier quasipossibilities.
  - *Numerical:* *n*xPx is true iff there are at least *n* (*n* > 1) individuals such that Pi_k (1≤ *k* ≤ *n*) is true.
  - *Proportional:* *m*/*n*xPx is true iff for at least *m*/*n*th (*m*/*n* ≤ ½) of the individuals i ∈ I, Pi is true.

Why *n* and *m*/*n* are quantifier quasipossibilities

- *n* and *m*/*n* satisfy the four quasimodal conditions.
  - I leave the demonstration of this as an exercise.
- *n* and *m*/*n* are duals of quasinecessity operators.
  - See next slide.

n and *m*/*n* are duals of quasinecessity operators

- Most of the numerical and proportional quasipossibility operators are duals of quasinecessities that are not idiomatically expressed in any language I am familiar with. However the quasipossibility *half of* (which expresses ½) is the dual of the quasinecessity *a majority of*.
- The quasipossibility quantifier *n* is the dual of the quasinecessity "all but fewer than *n*", which I represent $\forall^{-n}$. Consequently:
  - $\forall xPx \models^{q} \forall^{-n}xPx \models^{q} nxPx \models^{q} \exists xPx$
- The quasipossibility quantifier *m*/*n* is the dual of the quasinecessity "all but fewer than *m*/*n*", which I represent $\forall^{-m/n}$. Consequently:
  - $\forall xPx \models^{q} \forall^{-m/n}xPx \models^{q} m/nxPx \models^{q} \exists xPx$

Are there deontic quasimodals?

- Let $E^{\alpha d} = \langle S^{\alpha}, \models^{d}\rangle$ be the entailment structure previously defined for deontic modals.
  - What is the status of deontic operators such as those expressed by *have a duty to*, *be advised to*, *ought to*, *should*, etc.? Specifically, are they necessity or quasinecessity operators?
  - This is an empirical question. The answer depends on whether they distribute over multi-premise entailments.
    - In my judgment they do, which would render them necessity (modal) rather than quasinecessity (quasimodal) operators.
    - If this is correct, then epistemic *should* is a quasinecessity operator, but deontic *should* is a necessity one.

Interactions of epistemic quasimodals

- Interactions of like epistemic quasimodals (two quasinecessity or two quasipossibility operators)
  - $^{\diamond}⌂^{e}p\,^{\diamond}⌂^{e}p \Leftrightarrow^{e} {}^{\diamond}⌂^{e}p$ $\quad$ $^{\square}⌂^{e}p\,^{\square}⌂^{e}p \Leftrightarrow^{e} {}^{\square}⌂^{e}p$
    - *it is likely that it is likely that* p ⇔ᵉ *it is likely that* p
    - *it is not unlikely that it is not unlikely that* p ⇔ᵉ *it is not unlikely that* p
- Interactions of unlike epistemic quasimodals (a quasinecessity operator with a quasipossibility one)
  - $^{\diamond}⌂^{e}p\,^{\square}⌂^{e}p \models^{e} {}^{\square}⌂^{e}p\,^{\diamond}⌂^{e}p$ $\quad$ (instance of $^{\diamond}⌂$-over-$^{\square}⌂$)
    - *it is not unlikely that it is likely that* p $\models^{e}$ *it is likely that it is not unlikely that* p.
      - If p is neither likely nor unlikely, then the premise is false but the conclusion is true; otherwise both premise and conclusion have the same truth value.

Interactions of like quantifier quasimodals

- Let Ξ be a quasinecessity quantifier and Δ a quasipossibility quantifier.
  - $\Xi x\Xi yPxy \not\models^{q} \Xi y\Xi xPxy$, and conversely
  - $\Delta x\Delta yPxy \not\models^{q} \Delta y\Delta xPxy$, and conversely
    - *for most boys, they like most girls* $\not\models^{q}$ *for most girls, most boys like them*, and conversely
    - *there are two girls who pushed three boys* $\not\models^{q}$ *there are three boys who two girls pushed*, and conversely

Interactions of unlike quantifier quasimodals

- Let Ξ and Δ be as on the previous slide.
  - $\Delta x\Xi yPxy \models^{q} \Xi y\Delta\, xPxy$ $\quad$ (instance of $^{\diamond}⌂$-over-$^{\square}⌂$)
    - there are three boys that most girls like $\models^{q}$ for most girls, they like three boys
    - there are many boys that all but fewer than three girls like $\models^{q}$ for all but fewer than three girls, they like many boys

Interactions between epistemic and quantifier quasimodals

- Let $⌂^e$ be an epistemic quasimodal, and Π a quantifier quasimodal.
  - $\Pi x ⌂^e Px \not\models^q ⌂^e \Pi x Px$, and conversely
    - *there are three prisoners who it is (not un)likely will escape* $\not\models^q$ *it is (not un)likely that three prisoners will escape*, and conversely
    - for most prisoners it is (not un)likely that they will escape $\not\models^q$ *it is (not un)likely that most prisoners will escape*, and conversely
- This interaction does not appear to be predictable simply from the separate interaction patterns of epistemic and quantifier quasimodals.

Epistemic (E) and quantifier (Q) modal–quasimodal (M/Qm) interactions

- $⌂^e$ behaves like $\Box^e$ in interaction with $\Diamond^e$, but like $\Diamond^e$ in interaction with $\Box^e$.
  1. $\Diamond^e ⌂^e p \models^e ⌂^e \Diamond^e p$
  2. $⌂^e \Box^e p \models^e \Box^e ⌂^e p$
- Π behaves like ∀ in interaction with ∃, but like ∃ in interaction with ∀.
  1. $\exists x \Pi y Pxy \models^q \Pi y \exists x Pxy$
  2. $\Pi x \forall y Pxy \models^q \forall y \Pi x Pxy$
- These patterns represent a blend of $\Diamond$-over-$\Box$ with $^{\Diamond}⌂$-over-$^{\Box}⌂$.

Examples of E and Q M/Qm interactions

- E M/Qm interaction examples
  - *it is possible that it is likely that* p $\models^e$ *it is likely that it is possible that* p
  - *it is likely that it is certain that* p $\models^e$ *it is certain that it is likely that* p
- Q M/Qm interaction examples
  - *there is a boy who many girls like* $\models^q$ *there are many girls who like a boy*
  - *there are many girls who like every boy* $\models^q$ *for every boy there are many girls who like him or her*

Epistemic–Quantifier (E-Q) M/Qm interactions

- $⌂^e$/Π behaves like ∀ in interaction with $\exists/\Diamond^e$, and like ∃ in interaction with $\forall/\Box^e$.
  1. $\Diamond^e \Pi x Px \models^q \Pi x \Diamond^e Px$
  2. $\exists x ⌂^e Px \models^q ⌂^e \exists x Px$
  3. $⌂^e \forall x Px \models^q \forall x ⌂^e Px$
  4. $\Pi x \Box^e Px \models^q \Box^e \Pi x Px$
- These patterns also represent a blend of $\Diamond$-over-$\Box$ with $^{\Diamond}⌂$-over-$^{\Box}⌂$.

Examples of E-Q M/Qm interactions

- *it is possible that many prisoners will escape* $\models^q$ *there are many prisoners who it is possible will escape*
- *it is likely that every prisoner will escape* $\models^q$ *for every prisoner it is likely that he or she will escape*
- *there is a prisoner who it is likely will escape* $\models^q$ *it is likely that a prisoner will escape*
- *there are many prisoners who it is certain will escape* $\models^q$ *it is certain that many prisoners will escape*

Deontic–Quantifier (D-Q) M/Qm interactions

- Π behaves like ∀ in interaction with $\Diamond^d$, but like ∃ in interaction with $\Box^d$.
  - $\Diamond^d \Pi x P(a)x \models^q \Pi x \Diamond^d P(a)x$ $\qquad$ $\Pi x \Box^d P(a)x \models^q \Box^d \Pi x P(a)x$
- These patterns also represent a blend of $\Diamond$-over-$\Box$ with $^{\Diamond}⌂$-over-$^{\Box}⌂$.
- Examples:
  - *it is permitted for two girls to leave* $\models^q$ *there are two girls who are permitted to leave*
  - *it is permitted for most girls to leave* $\models^q$ *for most girls, they are permitted to leave*
  - *there are two girls who are required to leave* $\models^q$ *it is required for (any) two girls to leave*
  - *for most girls, they are required to leave* $\models^q$ *it is required for most girls to leave*

Deontic–Epistemic (D-E) M/Qm interactions

- Propositions with a deontic modal and epistemic quasimodal do not entail each other (cf. slide 29 for the same result for D-E modal interactions).
  1. ◇^{d}⌂^{e}Pa ⊭^{d} ⌂^{e}◇^{d}Pa, and conversely
  2. ⌂^{e}□^{d}Pa ⊭^{d} □^{d}⌂^{e}Pa, and conversely
- Examples:
  - *Ann is permitted to be (not un)likely to leave* ⊭^{d} *it is (not un)likely for Ann to be permitted to leave*, and conversely
  - *it is (not un)likely that Ann is required to leave* ⊭^{d} *Ann is required for it (not) to be (un)likely for her to leave*, and conversely

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Summary

1. Summary regarding modals
2. Summary regarding quasimodals

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Summary: Modals

1. Provided a structuralist account of necessity (□) and possibility (□) modals.
2. Stated a special modal law (□-over-□) governing the strength of scope relations of those operators in certain domains.
3. Described the following domains:
   a) E(pistemic modals) (obey □-over-□)
   b) Q(uantifier modals) (obey □-over-□)
   c) D(eontic modals) (do not obey □-over-□)
4. Described interactions of modals within and across domains.
   a) E-Q interactions obey □-over-□.
   b) D-Q interactions obey □-over-□.
   c) D-E interactions do not obey □-over-□.

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Summary: Quasimodals

1. Provided a structuralist account of quasimodals (⌂), which are weaker than □ and stronger than ◇.
   - Like □ and ◇, ⌂ distributes over single-premise entailments
   - Unlike □, ⌂ fails to distribute over multi-premise entailments.
   - Unlike ◇, ⌂ fails to distribute over disjunctions.
2. Distinguished two types of quasimodals: quasinecessities (□⌂) and quasipossibilities (◇⌂).
   - If ⌂ distributes over the law of contradiction, it is a □⌂.
   - The dual of □⌂ is ◇⌂.

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Summary: Quasimodals

3. Noted the following quasinecessity and quasipossibility operators.

Type	Quasinecessity	Quasipossibility
Epistemic	□⌂^{e}: *likely*	□⌂^{e}: *not unlikely*
Quantifier	M: *most*	μ: *many*
	∀$^{-n}$: all but <*n*	*n*: $n \geq 2$
	∀$^{-m/n}$: all but <*m/n*	*m/n*: $m/n \leq ½$
Deontic	(none)	(none)

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Summary: Quasimodals

4. Showed that the interactions of quasinecessity with quasipossibility operators obey □⌂-over-□⌂, except when epistemic and quantifier quasimodals interact, a fact for which no explanation was offered.
5. Showed that quasimodal-modal interactions obey a blend of □-over-□ with □⌂-over-□⌂, except as predicted, when epistemic quasimodals interact with deontic modals.

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