

LING 501, Fall 2004: Probability

Probability structures

The theory of probability is concerned with nonempty sets of **allowed situations** or **possibilities**. Given a set S of n **distinct** possibilities (**d-possibilities**), each member i of S is associated with a **probability** $p(i)$ of occurrence in the interval $0 < p(i) \leq 1$. Furthermore the $p(i)$ s sum to 1. For example the set of outcomes of a single toss of a balanced coin is $\{H, T\}$, where H represents the d-possibility that the coin lands head up, and T that the coin lands tail up, and $p(H) = p(T) = \frac{1}{2}$. D-possibilities can be combined using disjunction, and the probability of each resulting **composite possibility** (**c-possibility**) is the sum of the probabilities of its disjuncts. For example, the probability that the coin lands either heads up or tails up is $p(H|T) = p(H) + p(T) = \frac{1}{2} + \frac{1}{2} = 1$. A **probability structure** consists of a set of d-possibilities with their associated probabilities, and all the c-possibilities and their associated probabilities that can be formed from them. Figure 1 is a Hasse diagram for this simple coin toss probability structure. The c-possibility for which $p=1$ is the top (\top) of the structure. If we add to the structure the bottom (\perp) element, for which $p=0$, then the negation of $H = \top$ and vice versa, and the negation of $\top = \perp$ and vice versa. The probability of the negation of a possibility s with probability p is $1-p$.

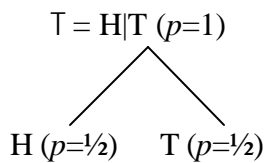


Figure 1. Probability structure for a single coin toss

Next consider a dollar slot machine that returns \$0 with $p = .7$, \$2 with $p = .2$, and \$5 with $p = .1$. I represent the three d-possibilities as $-\$1$ (the player loses one dollar), $+\$1$ (the player gains one dollar) and $+\$4$ (the player gains four dollars). The Hasse diagram for this structure is given in Figure 2.

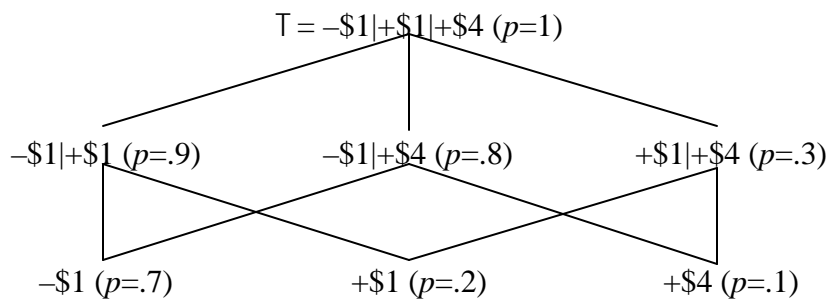


Figure 2. Probability structure for a single play of a slot machine

Figure 2 also shows that if two c-possibilities overlap, then their overlap is their conjunction. Therefore the probability of the conjunction of two overlapping c-possibilities is the probability of their overlap. The probability of the conjunction of nonoverlapping possibilities (e.g. of two d-possibilities) is 0.

A clearer example illustrating the conjunction of overlapping c-possibilities is given by the probability structure in which a balanced coin is tossed twice. Then the probability of the

conjunction of the c-possibility that the first toss lands head up (for which $p=1/2$) with the c-possibility that the second toss lands tail up (for which $p=1/2$) is the probability of their overlap, namely the d-possibility that the first toss lands head up and the second toss lands tail up, which is $1/4$. *Question:* Why is the first coin landing head up a c-possibility?

On the other hand, the probability of the disjunction of two overlapping possibilities is the sum of their probabilities minus the probability of their overlap, i.e. $p(s|t) = p(s) + p(t) - p(s&t)$. For example, the probability that a coin will land heads up on either of two tosses is given by $p(H1|H2) = p(H1)+p(H2)-p(H1\&H2) = 1/2+1/2-1/4 = 3/4$.

Conditional probability

Suppose in the slot machine example, we want to know the probability that if the player won (call that situation *A*), the player won \$4 (call that situation *B*). This is known as the **conditional probability** of *B* given *A*, usually symbolized $p(B|A)$. To avoid confusion with the disjunction operator, I represent the conditional probability as $p(B||A)$. This is distinct from $p(A\rightarrow B)$, which in this case is $p([+ \$1|+ \$4]\rightarrow + \$4) = .8$. To calculate $p(B||A)$, we can construct a new probability structure in which only winning situations are considered, and recalculate all the probabilities for the d- and c-possibilities in that structure, as in Figure 3. One way to recalculate the probabilities is to use **Bayes' theorem** according to which $p(B||A)$ is equivalent to $p(A||B)*p(B)/p(A)$. Since $p(A||B) = p(T) = 1$, the desired result is $p(+ \$4)/p([+ \$1|+ \$4]) = .1/.3 = 1/3$.

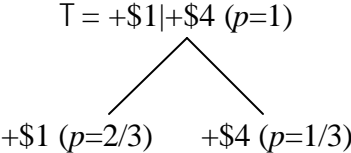


Figure 3. Probability structure derived from Figure 2 for winning situations only

Likelihood, certainty and possibility

Suppose that the proposition *Ls*, where *L* is the predicate *it is likely that* and *s* is a possibility of some probability structure, is true if and only if $p(s) > .5$. Then using the structure in Figure 1, *Ls* is false for both the d-possibilities. The propositions *it is likely that the coin will land head up* and *it is likely the coin will land tail up* are both false. On the other hand, using Figure 2, *Ls* is true for the d-possibility $-\$1$; the proposition *it is likely that the player will lose \$1* is true, since $p(-\$1) > .5$.

These two examples illustrate the following point: at most one d-possibility in any probability structure is likely. The reason is simple. If the probability of a particular d-possibility is greater than .5 (i.e. is likely), the probabilities of all the others are less than .5 (i.e. are not likely), since those probabilities must sum to unity.

A possibility is **certain** if its associated probability is 1. A d-possibility is certain if and only if it is the only d-possibility in the structure. Otherwise, only the disjunction of all the d-possibilities in the structure is certain. Thus in Figure 1, the proposition *it is certain that the coin will land head up or tail up* is true, but *it is certain that the coin will land head up* is false.

Every possibility in a probability structure is by definition **possible**. Hence all of the following propositions are true for the structure in Figure 1: *it is possible that the coin will land head up*; *it is possible that the coin will land tail up*; *it is possible that the coin will land head up or tail up*. So what's not possible? For one thing, the conjunction of d-possibilities: *it is possible that the coin will land tail up and head up* is false. Letting D represent the operator *it is possible that*, and s and t a pair of distinct d-possibilities, we find that:

- (1) $Ds \mid Dt \Leftrightarrow D[s \mid t]$
- (2) $D[s \ \& \ t] \models Ds \ \& \ Dt$, but not conversely.

Modal operators for probability

The relations in (1) and (2) are characteristic of **possibility modal operators**, which are conventionally symbolized by a diamond (\diamond). The converse of possibility is **necessity**. A **necessity modal operator** (conventionally symbolized by a box (\square)) satisfies the following laws (discussed further in the Modality handout).

- (3) $\square[s \ \& \ t] \Leftrightarrow \square s \ \& \ \square t$
- (4) $\square s \mid \square t \models \square[s \mid t]$, but not conversely.

I now show that *it is certain that* as applied to possibilities in a probability structure satisfies the relations in (3) and (4) and hence is a necessity modal operator under those conditions. First consider (3). Since there is exactly one (typically composite) possibility that is certain in any probability structure, namely \top , both sides of (3) are false whenever s and t are distinct possibilities. On the other hand, if s and t are the same, then both sides of (3) are true if $s = t = \top$ and false otherwise. Second consider (4), and suppose that neither s nor t is \top , but that $s \mid t = \top$. Then the left side of (4) is false and the right side is true. On the other hand, there is no condition under which the left side of (4) is true and the right side is false.