Introduction
Modality is the theory of necessity and possibility. Various ways of formulating that theory have been proposed, most famously by Saul Kripke making use of the theory of possible worlds. In this presentation, I basically follow the presentation in Koslow, A. (1992) A Structuralist Theory of Logic.

Necessity
Let $S$ be a nonempty set and let $\leq$ be an entailment relation on $S$, and let $\geq$ be its dual. Then a necessity operator $\Box$ (box) in the system $E$ consisting of the set $S$ and the relation $\leq$ satisfies the conditions in 1 and 2.

1. $\Box$ preserves entailment in $E$. That is for all $s_1, \ldots, s_n, t \in S$: if $s_1, \ldots, s_n \leq t$, then $\Box s_1, \ldots, \Box s_n \leq \Box t$
2. $\Box$ does not preserve entailment in $E'$, the dual of $E$. That is, for some $s_1, \ldots, s_n, t \in S$ for which $s_1, \ldots, s_n \geq t, \Box s_1, \ldots, \Box s_n \geq \Box t$ fails.

From 3, we conclude box both distributes over and factors out of conjunction, as in 3. From the failure of box to preserve entailment in $E'$, we also conclude that box factors out of but does not distribute over disjunction in $E$, as in 4.

3. $\Box [s & t] \leftrightarrow \Box s & \Box t$ for every $s, t \in S$
4. $\Box s | \Box t \leq \Box [s | t]$ for every $s, t \in S$, but $\Box [s | t] \leq \Box s | \Box t$ fails for some $s, t \in S$.

For example, if $S$ is a set properly including $T$ and $\bot$, then the operator $\Box^l$ defined in 5 is a necessity operator. This operator is called logical necessity.

5. For all $s$, if $\leq s$, then $\leq \Box^l s$; otherwise $\Box^l s \leq$

Exercise 1
Verify that $\Box^l$ satisfies the conditions in 3 and 4.

Possibility
The dual of necessity is possibility, symbolized $\Diamond$ (diamond). The possibility operator satisfies the conditions in 6 and 7.

6. $\Diamond s | \Diamond t \leftrightarrow \Diamond [s | t]$ for every $s, t \in S$.
7. $\Diamond [s & t] \leq \Diamond s & \Diamond t$ for every $s, t \in S$, but $\Diamond s & \Diamond t \leq \Diamond [s & t]$ fails for some $s, t \in S$.

For example, if $S$ is a set properly including $T$ and $\bot$, then the operator $\Diamond^l$ defined in 8 is a possibility operator. This operator is called logical possibility and may also be represented $\Diamond^l$.

8. For all $s$, if $\leq s$, then $\leq \Diamond^l s$; otherwise $\leq \Diamond^l s$.

Necessity and possibility operators are known as modal operators.
Interdefinability of $\square$ and $\Diamond$ using the not-not equivalences

Modal operators are generally interdefinable using the not-not equivalences in 9 and 10. The second column results from negating both sides of the equivalences in the first column and canceling double negation.

9.  $\square s \leftrightarrow \neg \neg \Diamond \neg s$  \quad $\neg \square s \leftrightarrow \Diamond \neg s$

10. $\Diamond s \leftrightarrow \neg \neg \Box \neg s$  \quad $\neg \Diamond s \leftrightarrow \Box \neg s$

Some other common properties of modal operators

Some necessity operators satisfy additional conditions such as 11 and 12. To show that $\square^I$ satisfies 11, we observe that if $s$ is a tautology then $\square^I s$ is a tautology, which satisfies 11. Otherwise $\square^I s$ is a contradiction so again 11 is again satisfied. A similar argument shows that $\square^I$ satisfies 12.

11. $\square s \leq s$

12. $\square s \leq \Box \Box s$

Some possibility operators satisfy conditions 13 and 14, which correspond to 11 and 12. Note that by transitivity of entailment, we obtain from 11 and 13 the result that $\square s \leq \Diamond s$.

13. $s \leq \Diamond s$

14. $\Diamond \Diamond s \leq \Diamond s$

Exercise 2

Show that that $\Diamond^I$ satisfies 13 and 14, and also that $\square^I$ and $\Diamond^I$ together satisfy condition 15.

15. $\Diamond s \leq \Box \Diamond s$

Epistemic necessity and possibility

Let $|=\mathcal{S}$ be truth-preservation entailment over a set $\mathcal{S}$ of propositions such that for every $P \in \mathcal{S}$, there are propositions $\Box^e P$ ‘it is certain that $P$’ and $\Diamond^e P$ ‘it is not certain that not $P$’. We may call $\Box^e$ epistemic necessity and $\Diamond^e$ epistemic possibility.

Exercise 3

Show that $\Box^e$ is a necessity operator and that $\Diamond^e$ is a possibility operator.

Exercise 4

Show that 16 and 17 hold for every $P \in \mathcal{S}$. Under what condition do the reverse entailments (in which the premises and the conclusions are interchanged) fail?

16. $\square^I P |= \square^e P$

17. $\Diamond^e P |= \Diamond^I P$
Relativized epistemic necessity

Corresponding to the proposition $\□^e P$, let there be a proposition $K(i, P) \in S$, where $i$ is an individual that stands in a certainty relation to $P$; $K(i, P)$ may be glossed ‘$i$ is certain that $P$’. Neither proposition entails the other, since $i$ may be certain of propositions that are not epistemically necessary, and $i$ may not be certain of propositions that are epistemically necessary. Nevertheless the derived operator $\square^i$, may be considered a relativized epistemic necessity operator, representable as $\square^e^i$, if the modal laws 1 and 2 hold for it.

Presumptive necessity and possibility

Again let $|=\!$ be truth-preservation entailment over a set $S$ of propositions such that for every $P \in S$, there are propositions $\Box^p P$ ‘it is presumed that $P$’ and $\Diamond^p P$ ‘it is not presumed that not $P$’. We may call $\Box^p$ presumptive necessity, and $\Diamond^p$ presumptive possibility. Every epistemically necessary proposition is also presumptively necessary, but if some propositions that are not certain are presumed, then the converse is not true. Moreover the modal laws 11 and 13 do not hold for presumptive necessity, since $\Box^p P$ can be true even though $P$ is false, and $\Diamond^p P$ false even though $P$ is true.

Presumptive necessity and possibility are also relativizable to an individual $i$, and the sentence It must have rained yesterday expresses presumptive necessity relativized to the speaker, not epistemic necessity as is usually supposed.

Summary so far

For any entailment system in which logical, epistemic and presumptive necessity and possibility are defined, those operators may be diagrammed as in Figure 1.

\begin{figure}[h]
\centering
\begin{tikzpicture}[level distance=1.5cm,sibling distance=2.5cm,auto]
  \node {$\Diamond^p$}
  child {node {$\Box^p$}
    child {node {$\Box^e$}
      child {node {$\Diamond^l$}}
      child {node {$\Diamond^e$}}
    }
    child {node {$\Diamond^e$}}
  }
  child {node {$\Diamond^p$}}
\end{tikzpicture}
\caption{Entailment relations among the various modal operators defined so far.}
\end{figure}

Deontic necessity and possibility

Let $|=\!$ be truth-preservation entailment over a set $S$ of propositions involving actions that can be performed by an individual. Then for every such proposition $P$ and individual $i$ in the domain, there may be two propositions: $\Box^d P i$ ‘$i$ is obligated to do $P$’, and $\Diamond^d P i$ ‘$i$ is permitted to do $P$’. We
may call □₅ deontic necessity, and ◇₅ deontic possibility. Like presumptive necessity and possibility, deontic necessity and possibility do not obey the modal laws 11 and 13, since one may be obligated to do what one is not permitted to do; one may fail to do what one is obligated to do and one may do what one is not permitted to do. Moreover, there do not appear to be any entailment relations between deontic necessity and possibility and the other modal operators defined so far.

**Abilitative necessity and possibility**

Deontic necessity expresses an obligation imposed on an individual by another, and deontic possibility the freedom from the corresponding negative obligation. However, an obligation can also be imposed on an individual as it were from within, as in the interpretation of the multiply ambiguous sentence 18 in which Alex is understood as internally compelled to eat. I interpret this necessity as abilitative necessity, symbolized □ₕ, since it paraphrases 19, in which the abilitative possibility operator, symbolized ◇ₕ, appears in the *not-not* construction.

18. Alex has to eat.
19. Alex isn't able not to eat.

**Circumstantial necessity and possibility**

Example 18 has another interpretation, in which an obligation is imposed on Alex, not by an external agent nor by Alex herself, but by circumstances, such as that Alex has not eaten for three days. There is no agreed-upon name for this modality to my knowledge; I call it circumstantial necessity and represent it □ₖ. Its circumstantial possibility counterpart ◇ₖ denotes freedom from circumstantial necessity.